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Que (1) :- State and prove De Morgan's and Bertrand's Test.

Ans :- A Series $\sum u_n$ of positive terms is Convergent or divergent according as

$$\lim_{n\to\infty} \left[\left\{ n\left(\frac{u_n}{u_{n+1}} - 1 \right) - 1 \right\} \log n \right] > 1 \text{ or } < 1$$

Let us Compare the given Series $\sum u_n$ with the auxiliary Series $\sum v_n = \sum \frac{1}{n(\log n)^p}$ So that $v_n = \frac{1}{n(\log n)^p}$

We know that the Series $\sum v_n$ is Convergent if $p > 1$ and divergent if $p \geq 1$

Now $\frac{v_n}{v_{n+1}} = \frac{1}{n(\log n)^p} \div \frac{1}{(n+1)\{\log(n+1)\}^p}$

$$= \frac{(n+1)\{\log(n+1)\}^p}{n(\log n)^p}$$

$$= \left(\frac{n+1}{n}\right) \cdot \left\{\frac{\log(n+1)}{\log n}\right\}^p$$

$$= \left(1+\frac{1}{n}\right) \left\{\frac{\log n(1+1/n)}{\log n}\right\}^p$$

$$= \left(1+\frac{1}{n}\right) \left\{\frac{\log n + \log(1+\frac{1}{n})}{\log n}\right\}^p$$

$$= \left(1+\frac{1}{n}\right) \left\{\frac{\log n + \frac{1}{n} - \frac{1}{2n^2} + \cdots}{\log n}\right\}^p$$

$$= \left(1+\frac{1}{n}\right) \left\{1 + \frac{1}{n\log n} - \frac{1}{2n^2\log n} + \cdots\right\}^p$$

$$= \left(1+\frac{1}{n}\right) \left\{1 + \frac{1}{n\log n} - \frac{1}{2n^2\log n} + \cdots\right\}^p$$

$$= \left(1+\frac{1}{n}\right) \left\{1 + \frac{p}{n\log n} + \cdots\right\}$$

$= 1 + \frac{1}{n} + \frac{p}{n\log n} + \cdots$ terms Containing higher powers of $\frac{1}{n}$.

Case I :— Let ΣV_n be Convergent i.e $p>1$ then ΣU_n will also be Convergent if

$$\frac{U_n}{U_{n+1}} > \frac{V_n}{V_{n+1}}$$

i.e if $\frac{U_n}{U_{n+1}} > 1 + \frac{1}{n} + \frac{p}{n\log n} + \cdots$

i.e if $\frac{U_n}{U_{n+1}} - 1 > \frac{1}{n} + \frac{p}{n\log n} + \cdots$

i.e if $n\left(\frac{U_n}{U_{n+1}} - 1\right) > 1 + \frac{p}{\log n} + \cdots$

i.e if $\left\{n\left(\frac{U_n}{U_{n+1}} - 1\right) - 1\right\}\log n > p + \cdots$

terms Containing $\frac{1}{n}$ and higher Powers of $\frac{1}{n}$.

i.e if $\lim_{n\to\infty}\left[\left\{n\left(\frac{U_n}{U_{n+1}} - 1\right) - 1\right\}\log n\right] \geq p (>1)$

But $p>1$. Hence ΣU_n is Convergent if

$$\lim_{n\to\infty}\left[\left\{n\left(\frac{U_n}{U_{n+1}} - 1\right) - 1\right\}\log n\right] > 1$$

Case II :— Let ΣV_n be divergent i.e $p \leq 1$

Then ΣU_n will also be divergent

$$\frac{U_n}{U_{n+1}} < \frac{V_n}{V_{n+1}}$$

i.e if $\lim_{n\to\infty}\left[\left\{n\left(\frac{U_n}{U_{n+1}} - 1\right) - 1\right\}\log n\right] \leq p (<1)$

proceeding exactly as in Case I above.

i.e if $\lim_{n\to\infty}\left[\left\{n\left(\frac{U_n}{U_{n+1}} - 1\right) - 1\right\}\log n\right] < 1$

Since $p<1$

Hence proved